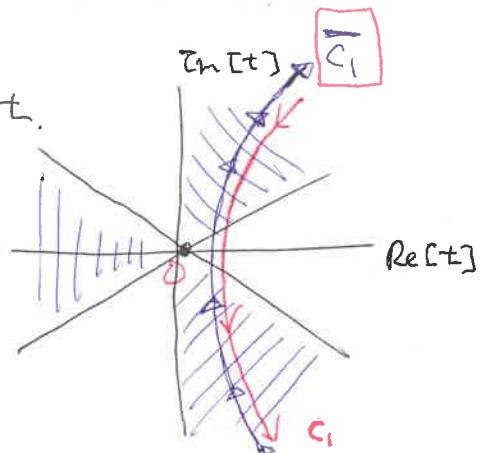


② $z \rightarrow -\infty$: Let $S = -|z|^{\frac{1}{2}} +$ (inversion w.r.t. \mathbb{D})

$$Ai(z) = \frac{1}{2\pi i} |z|^{\frac{1}{2}} \int_{C_1} e^{|z|^{\frac{3}{2}}(t + \frac{t^3}{3})} dt.$$

$$= \frac{1}{2\pi i} |z|^{\frac{1}{2}} \int_{\underline{C_1}} e^{|z|^{\frac{3}{2}}(t + \frac{t^3}{3})} dt$$



→ Saddle point approx.

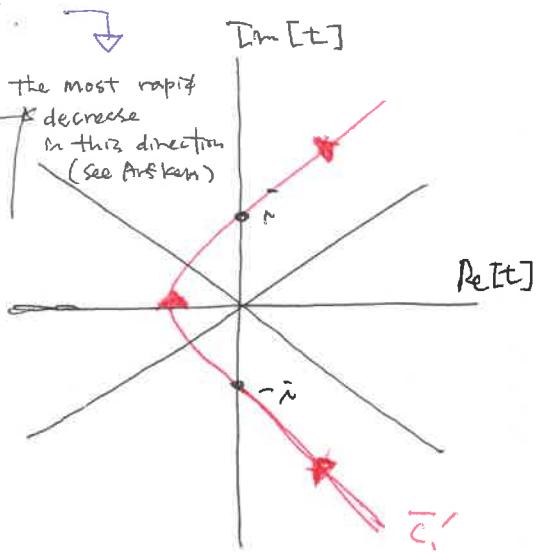
(two points : $t = \pm n$
equivalent)

→ Deform \bar{C}_1 to cross $t = \pm n$. \downarrow

$$\begin{aligned} t &= n + \sqrt[3]{n} \\ &\parallel \Gamma_n = e^{\frac{\pi i}{4}} = \frac{1+i}{\sqrt{2}} \end{aligned}$$

The most rapid decrease in this direction (See Arfken)

$$\begin{aligned} t + \frac{t^3}{3} &\approx \frac{2}{3}n^{\frac{3}{2}} - \frac{1}{3}n^2 \quad \text{around } t = +n \\ &\approx -\frac{2}{3}n^{\frac{3}{2}} - \frac{1}{3}n^2 \quad \text{around } t = -n \end{aligned}$$



$$Ai(z) \approx \frac{1}{2\pi i} |z|^{\frac{1}{2}} \left[e^{\frac{\pi i}{4}n} \int_{t+} d\zeta e^{|z|^{\frac{3}{2}}(\frac{2}{3}\zeta^{\frac{3}{2}} - \frac{1}{3}\zeta^2)} - e^{-\frac{\pi i}{4}n} \int_{t-} d\zeta e^{|z|^{\frac{3}{2}}(-\frac{2}{3}\zeta^{\frac{3}{2}} - \frac{1}{3}\zeta^2)} \right]$$

approx. $\int_{t\pm} d\zeta \rightarrow \int_{-\infty}^{\infty} d\zeta$ (integrand is only important near $\zeta = 0$)

$$\approx \frac{1}{2\pi i} |z|^{-\frac{1}{4}} \left[e^{i(\frac{2}{3}|z|^{\frac{3}{2}} + \frac{\pi}{4})} - e^{-i(\frac{2}{3}|z|^{\frac{3}{2}} + \frac{\pi}{4})} \right]$$

$$= \frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}} \sin \left[\frac{2}{3}|z|^{\frac{3}{2}} + \frac{\pi}{4} \right]$$

↳ on

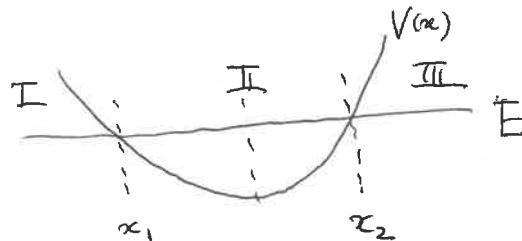
$$Ai(z) \approx \frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}} \cos \left[\frac{2}{3}|z|^{\frac{3}{2}} - \frac{\pi}{4} \right] \quad \text{as } z \rightarrow -\infty$$

Similarly,

$$\text{Bi}(z) \approx \begin{cases} \frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}} e^{\frac{2}{3}|z|^{\frac{3}{2}}} & \text{as } z \rightarrow \infty \\ -\frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}} \sin\left(\frac{2}{3}|z|^{\frac{3}{2}} - \frac{\pi}{4}\right) & \text{as } z \rightarrow -\infty \end{cases}$$

← unphysical!

Now, coming back to



$$\text{I : } U_E(x) \sim \exp\left[-\frac{1}{\hbar} \int_x^{x_1} dx' \sqrt{2m[V(x')-E]}\right]$$

$$\text{near } x_1, \quad V(x) \simeq E + |V'(x_1)| (x_1 - x) + \dots$$

$$\Rightarrow \frac{1}{\hbar} \int_x^{x_1} dx' \sqrt{2m(x_1 - x') |V'(x_1)|} = \sqrt{\frac{2mV'}{\hbar^2}} \cdot \frac{2}{3} (x_1 - x)^{\frac{3}{2}}$$

$$= \frac{2}{3} Z^{\frac{3}{2}} \quad \parallel Z = \left(\frac{2mV'}{\hbar^2}\right)^{\frac{1}{3}} (x_1 - x)$$

$$\therefore U_E(x) \sim A_i(z) \quad \text{as } z \rightarrow \infty$$

$$\text{III : } U_E(x) \sim \exp\left[-\frac{1}{\hbar} \int_{x_2}^x dx' \sqrt{2m[V(x')-E]}\right]$$

$$\text{near } x_2, \quad V(x) \simeq E + |V'(x_2)| (x - x_2) + \dots$$

$$\Rightarrow \frac{1}{\hbar} \int_{x_2}^x dx' \sqrt{2m|V'(x_2)|(x - x_2)} = \sqrt{\frac{2mV'}{\hbar^2}} \cdot \frac{2}{3} (x - x_2)^{\frac{3}{2}}$$

$$= \frac{2}{3} Z^{\frac{3}{2}} \quad \parallel Z = \left(\frac{2mV'}{\hbar^2}\right)^{\frac{1}{3}} (x - x_2)$$

$$\therefore U_E(x) \sim A_i(z) \quad \text{as } z \rightarrow \infty$$

$\text{II} \circledast$ ① from x_1 ,

$$U_E(x) \sim \cos \left[\frac{1}{\hbar} \int_{x_1}^x dx' \sqrt{2m(E-V(x'))} \right] - \frac{\pi}{4}$$

$$\text{def. of } z \text{ in I} \rightarrow \frac{2}{3} |z|^{\frac{3}{2}}$$

from $A_i(z)$ as $z \rightarrow \infty$

② from x_2

$$U_E(x) \sim \cos \left[\frac{1}{\hbar} \int_x^{x_2} dx' \sqrt{2m(E-V(x'))} \right] - \frac{\pi}{4}$$

$$\text{def. of } z \text{ in III} \rightarrow \frac{2}{3} |z|^{\frac{3}{2}}$$

from $A_i(z)$ as $z \rightarrow -\infty$

Phase Matching For ALL x

$$\frac{1}{\hbar} \int_{x_1}^x dx' [] - \frac{\pi}{4} = - \frac{1}{\hbar} \int_x^{x_2} dx' [] + \frac{\pi}{4} + n\pi$$

ans!

$\Leftrightarrow \cos \theta = \cos(-\theta)$.

\Rightarrow Quantization condition (WKB)

$$\int_{x_1}^{x_2} dx' \sqrt{2m[E-V(x')]} = \left(n + \frac{1}{2}\right) \pi \hbar$$

ans

\leftrightarrow Sommerfeld - Wilson quantization
(old quantum theory)

$$\oint p dx = nh.$$

†

$$\parallel P_{\text{cl.}} = \sqrt{2m(E-V)}$$

- Why is it "semi-classical"?

(MORE RIGOROUSLY...)

- Hamilton-Jacobi Theory

$$\text{Classical Action} \quad S(b, a) = \int_{t_a}^{t_b} dt \underline{\underline{L}}(q(t), \dot{q}(t), t)$$

Where $b \equiv q_b \equiv (q_1(t_b), q_2(t_b), \dots q_f(t_b))$

$a \equiv q_a \equiv (q_1(t_a), q_2(t_a), \dots q_f(t_a))$

def. Integral of L along the Trajectory

allowed by the eqs. of motion It's a physical trajectory.

from configuration q_a at t_a to q_b at t_b .

Taking an arbitrary variation of q

$$\delta S(b, a) = \int_{t_a}^{t_b} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \stackrel{=0}{\cancel{=}} \text{: We're on a physical trajectory!}$$

* NOTE:

$$\int_t^b \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i dt \\ \text{int. by parts}$$

$$+ \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{t_a}^{t_b}$$

$$\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \Big|_t - \int_t^b \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i dt$$

$\cancel{+0} \stackrel{\neq 0}{\sim} \text{: We do have variations at the end points!}$

$$\Rightarrow \delta S(b, a) = P_{b,i} \delta q_i(t_b)$$

$$- P_{a,i} \delta q_i(t_a)$$

$$\Rightarrow P_{b,i} = \frac{\partial}{\partial q_{b,i}} S(b, a) \quad , \quad P_{a,i} = - \frac{\partial}{\partial q_{a,i}} S(b, a)$$

$$= P_{b,i}(b, a)$$

$$= P_{a,i}(b, a)$$

Fix (q_a, t_a) ,

Consider the change of action in t_b

$$S(b, a) = \int_{t_a}^{t_b} dt L(q, \dot{q}; t)$$

$$\Rightarrow \frac{dS}{dt_b} = L(t_b)$$

$$\hookrightarrow \frac{\partial S}{\partial t_b} + \frac{\partial S}{\partial q_{b,i}} \dot{q}_{b,i} = \frac{\partial S}{\partial t_b} + P_{b,i} \dot{q}_{b,i}$$

Thus,

$$\frac{\partial S}{\partial t_b} + \underbrace{P_{b,i} \dot{q}_{b,i}}_{\equiv H(t_b)} - L(t_b) = 0$$

Then.

$$S = S(q_1, \dots, q_f, t)$$

For

$$H = H(q_1, \dots, q_f, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_f}; t).$$

$$\Rightarrow \boxed{\frac{\partial S}{\partial t} + H = 0} \Rightarrow \frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial q} \right)^2 + V(q) = 0$$

Hamilton-Jacobi equation

• For a time-indep. Hamiltonian, ($H = E$)

$$S(q, t) = W(q) - Et \quad \begin{array}{l} \text{Integrate} \\ \frac{\partial S}{\partial t} + H = 0 \end{array}$$

\uparrow Hamilton's characteristic function.

• If we consider a free particle in a potential

$$|\nabla S| = |\nabla W| = \sqrt{2m(E-V)}$$

or: If $V=0$.

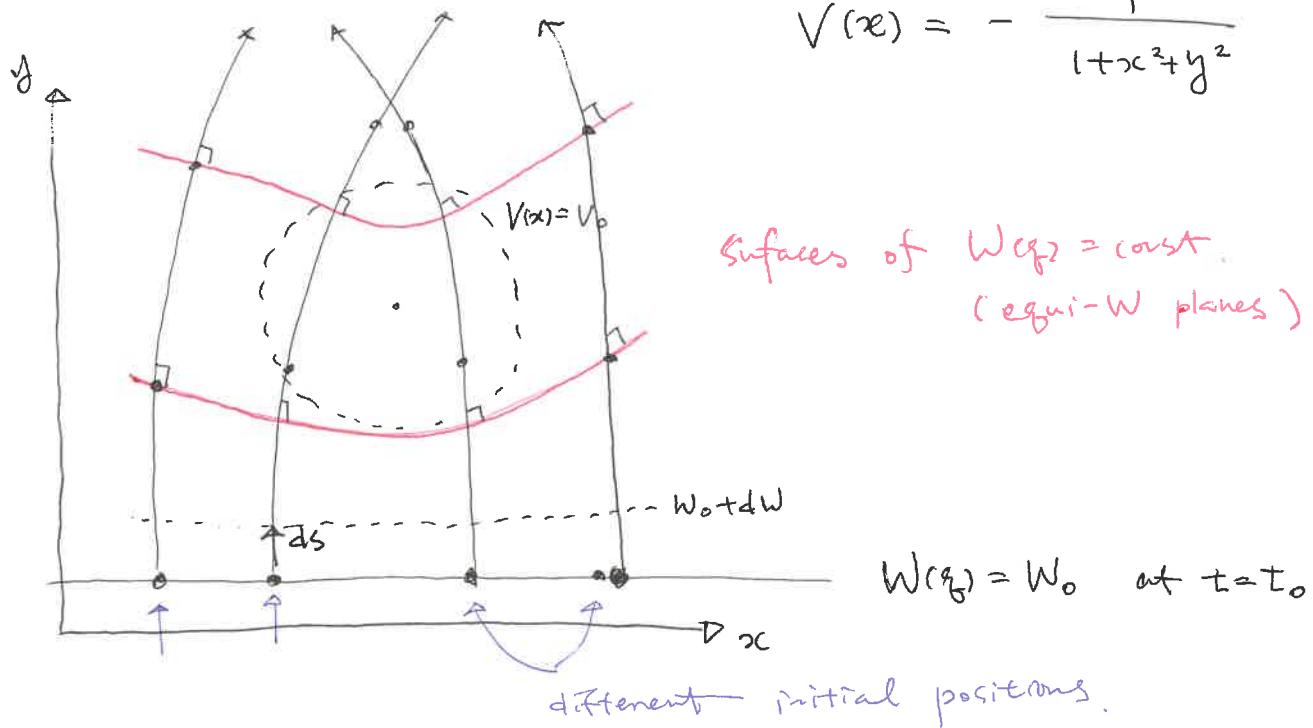
\therefore momentum.

$$W(x) \sim p^2 c$$

$$S(x, t) \sim p^2 c - Et$$

* meaning of S and W .

consider a particle moving in a potential



Along the trajectory, length ds for $W_0 \rightarrow W_0 + dW$

$$\Rightarrow ds = \frac{dW}{|\nabla W|} = \frac{dW}{\sqrt{2m[E - V_0]}} \quad | V_0 \equiv V(x_0)$$

The motion of $S(x_0, t) = \text{constant}$, $(dW = E dt)$
 $\therefore dS = 0$.

$$\Rightarrow \frac{ds}{dt} = \frac{E}{\sqrt{2m[E - V_0]}}$$

It's like
 $S \sim px - Et$.

If the surfaces of $S = \text{const.}$ as wave-fronts,

$$\frac{ds}{dt} = \underline{\text{phase velocity}}$$

$$\text{Ex. } e^{i(hx - Et)}$$

$$\Rightarrow v = \frac{dx}{dt} = \frac{\omega}{k}$$

\Rightarrow Classical Mechanics

\approx Geometrical optics.

(particles are not on the surface)
(wave fronts)

→ Semi classical interpretation of the Wave function

(Q.M.)

- From Hamilton-Jacobi Theory (C.M.)

$S \approx \text{phase factor of wave fn.}$

- Similarity to WKB approx.

HJ eq. for $H = \frac{p^2}{2m} + V(x)$ (t-indep.)

$$\frac{\partial S}{\partial t} + H = 0 \quad \text{with} \quad S = W(x) - Et$$

$$\Rightarrow -E + \frac{1}{2m} \left(\frac{\partial W}{\partial x} \right)^2 + V(x) = 0.$$

... HJ eq.

$$\Leftrightarrow - \left(\frac{\partial W}{\partial x} \right)^2 + \left[\frac{2m}{\hbar^2} (E - V(x)) \right] \cdot \hbar^2 = 0 \quad (\text{C.M.})$$

→ This is just WKB !

where $U_E(x) = e^{iW(x)/\hbar}$

Q.M.

⇒ Brillouin-Wentzel Ansatz

$$\Psi(x,t) = \exp \left[i \underline{\Theta(x,t)} / \hbar \right]$$

nos

Schrödinger eq.



$$\frac{\partial \Theta}{\partial t} + \frac{1}{2m} \left(\frac{\partial \Theta}{\partial x} \right)^2 + V(x) = \frac{i\hbar}{2m} \frac{\partial^2 \Theta}{\partial x^2}$$

\hbar is here only.

$$\boxed{\left(\frac{\partial S}{\partial x} \right)^2 \gg \hbar \left| \frac{\partial S}{\partial x^2} \right|}$$



Without this term, the Schrödinger eq.

as in

WKB.

→ just the HJ eq.

Thus, we may just write down

$$\Psi \sim \exp \left[\frac{i}{\hbar} \cdot \text{Classical Action} \right] \quad \text{when } t \rightarrow 0.$$

To see the quantum corrections, consider

$$\Theta = S + \frac{\hbar}{i} S_1 + \left(\frac{\hbar^2}{i} \right)^2 S_2 + \dots$$

↑
classical.

$$\Rightarrow O(\hbar) : \frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial x} \right)^2 + V(x) = 0$$

—————
H-J eq.

$$O(\hbar') : \frac{\partial S_1}{\partial t} + \frac{1}{2m} \left[2 \cdot \frac{\partial S}{\partial x} \cdot \frac{\partial S_1}{\partial x} + \frac{\partial^2 S}{\partial x^2} \right] = 0.$$

→ Find S_1 for a give S .

=D. Semi classical wave function

$$\Psi_{sc}(x,t) = \sqrt{P(x,t)} e^{i \frac{S(x,t)}{\hbar}}$$

wave

where

$$P(x,t) = |\Psi_{sc}|^2 = e^{2S_1(x,t)}$$

prob. density of finding a particle
near x at t .

(Max Born)

* Remark

"classical" means
here when $\Theta \rightarrow S$.
for a very small S_1 ,
H-J eq. is recovered

when

$$\left(\frac{\partial S}{\partial x} \right)^2 \gg \hbar \left(\frac{\partial^2 S}{\partial x^2} \right)$$

short wave length

(See WKB.)

Rewriting $O(\hbar')$ terms from $\rho = e^{2S}$,

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x} \left[\frac{1}{m} \cdot \frac{\partial S}{\partial x} \cdot \rho(x,t) \right] = 0.$$

using the current density $\vec{J}(x,t)$

$$\vec{J}(x,t) = \frac{e}{m} T_m \underbrace{[\psi^* \nabla \psi]}_{\psi \nabla \psi + \frac{i}{\hbar} e \psi} = \frac{e \cdot \nabla S}{m},$$

$\psi \nabla \psi + \frac{i}{\hbar} e \psi$

Thus, $O(\hbar')$ terms gone.

$$\int d^3 \vec{J} = \frac{\langle \vec{P} \rangle}{m}.$$

(a single particle!)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0. \quad \text{continuity eq.}$$

: prob. \propto conserved!
flux.

It just behaves
like a "flux".

You can directly show this

from the Schrödinger eq., also.

Hamilton - Jacobi equation

particle

particle - wave
duality

wave

as $t \rightarrow 0$, short wavelength

"geometrical
optics"

conceptually ...

Schrödinger equation

wave